Linear Algebra II

20/03/2012, Tuesday, 14:00-16:00

1 (20 pt) **Gram-Schmidt process**

Consider the vector space of $C[-1,1]$ with the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

Find the best approximation of the function $f(x) = x$ within the subspace $\text{span}\{1, |x|\}$.

2 (30 pt) **Eigenvalues**

Let $A \in \mathbb{R}^{n \times n}$ be a skew-symmetric matrix, that is $A = -A^T$.

(a) Show that $x^T A x = 0$ for each $x \in \mathbb{R}^n$ and $\text{Re}(x^H A x) = 0$ for each $x \in \mathbb{C}^n$.

(b) Show that $\text{Re}(\lambda) = 0$ if λ is an eigenvalue of A.

(c) Show that any skew-symmetric matrix is diagonalizable by a unitary matrix.

(d) Suppose that A is also an orthogonal matrix. Find all eigenvalues of A. (Hint: Prove first that if λ is an eigenvalue of A then $p(\lambda)$ is an eigenvalue of $p(A)$ where p is a polynomial.)

3 (20 pt) **Positive definite matrices**

(a) Let

$$A = \begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix}$$

where a and b are real numbers. Plot the region of (a,b)-plane in which A is positive definite. (Warning: The matrix A is not necessarily symmetric!)

(b) Consider the function

$$f(x,y) = \frac{x}{y^2} + \frac{y}{x^2} + xy + 1.$$

(i) Show that $(1,1)$ is a stationary point.

(ii) Determine the nature (local minimum, maximum, or saddle) of this stationary point.

4 (20 pt) **Singular value decomposition**

Find a singular value decomposition for the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

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10 pt gratis